M Math Analysis of several variables Midterm examination 14-09-2017.

Answer all the 7 questions. Each question is worth 6 points. Maximum score: 40

If you are using any result proved in the class, you need to state it correctly. If the answer is an immediate consequence of a result quoted, then that result also needs to be proved.

1. Let $T : R^n \to R^m$ be a one-to-one linear map. Show that there exists a $k > 0$ such that $\|T(x)\| \geq k\|x\|$ for all $x \in R^n$.
2. Let $f, g : R^n \to R$ be differentiable functions. Define $F : R^n \to R^2$ by $F((x_1, ..., x_n)) = (f(x_1, x_2, 0, ..., 0), g(0, 0, x_3, x_4, ..., x_n))$. Show that F is differentiable.
3. Let $f : R^3 \to R$ be a continuous function. Suppose $\frac{1}{f}$ is defined and differentiable in the open ball $B(0, 1)$. Is f differentiable? Justify your answer.
4. Let $f : R^2 \to R$ be defined by $f(x, y) = 0$ if $xy = 0$ and $f(x, y) = 1$ otherwise. Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exists at $(0, 0)$ but f is not continuous at $(0, 0)$.
5. Give an example with full details of a function $f : R^2 \to R$ for which $D_{12}f$ and $D_{21}f$ exists but are not equal.
6. For the functions $f, g : R^2 \to R$ defined by $f(x, y) = x^2 + y^2$, $g(x, y) = (x-1)^3 - y^2$. Show that the conclusion of Lagrange multiplier theorem fails to hold while solving for a minimum of f on the zero set of g.
7. Let $f : \{(x, y) \in R^2 : x + y < 1\} \to R$ be a convex function. Show that $F : [0, 1] \to R$ defined by $F(t) = f(\frac{1}{2}(1-t), \frac{t}{2})$ for $t \in [0, 1]$ is a convex function.